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HOMOLOGY OF FREE LIE POWERS AND TORSION IN GROUPS

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ABSTRACT

Let G be a group that is given by a free presentation $G = F/R$, and let $\gamma_4 R$ denote the fourth term of the lower central series of R. We show that if G has no elements of order 2, then the torsion subgroup of the free central extension $F/[\gamma_4 R, F]$ can be identified with the homology group $H_6(G, \mathbb{Z}/2\mathbb{Z})$. This is a consequence of our main result which refers to the homology of G with coefficients in Lie powers of relation modules.

1. Introduction

For an (additively written) free abelian group A, let LA denote the free Lie ring on A. Thus if $\mathcal{A}$ is a free $\mathbb{Z}$-basis of A, LA is the free Lie ring on the set $\mathcal{A}$, and the $\mathbb{Z}$-span of $\mathcal{A}$ in LA can be identified with A. The nth free Lie power $L^n A$ of A is the $\mathbb{Z}$-span of all left-normed Lie products $[a_1, a_2, \ldots, a_n]$ $(a_1, \ldots, a_n \in A)$ in LA. Let G be a group and suppose that the free abelian group A is a right G-module. Then the G-action on A extends uniquely to a G-action on LA turning the Lie powers $L^n A$ into G-modules (for $g \in G, [a_1, \ldots, a_n]g = [a_1 g, \ldots, a_n g]$). Now suppose that G is given by a free presentation.

$$(1.1) \qquad 1 \to R \to F \xrightarrow{\pi} G \to 1,$$

where F is a non-cyclic free group, and let $R_{\mathrm{ab}} = R/R'$ denote the relation module stemming from (1.1). Thus R_{ab} is the abelianized normal subgroup R regarded as a right G-module via conjugation in F. The homology $H_*(G, L^n R_{\mathrm{ab}})$ of G with coefficients in the Lie power $L^n R_{\mathrm{ab}}$ $(n \geq 2)$ has been studied in [12],

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[4]. This work was mainly motivated by the group theoretic relevance of the zero dimensional homology group $H_0(G, L^n R_{\mathrm{ab}})$. Let $\gamma_n R$ denote the nth term of the lower central series of R. Then there is an isomorphism

$$H_0(G, L^n R_{\mathrm{ab}}) \cong \gamma_n R/[\gamma_n R, F], \tag{1.2}$$

which enables us to interpret results on homology of Lie powers in purely group theoretic terms. The quotient on the right hand side of (1.2) is the kernel of the free central extension

$$1 \to \gamma_n R/[\gamma_n R, F] \to F/[\gamma_n R, F] \to F/\gamma_n R \to 1. \tag{1.3}$$

While $F/\gamma_n R$ is always torsion-free, elements of finite order may occur in the kernel of (1.3), even when $G = F/R$ is torsion-free. This was first discovered by C.K. Gupta [2] for the special case when $n = 2$ and $R = F'$. Since then, torsion in the free central extension (1.3) has been studied in a number of papers. We mention the pioneering work of Yu. V. Kuz'min [7], [8] on the case $n = 2$, and refer to [10], [3] for a detailed survey of these matters.

It is known that for $k \geq 1$ the homology group $H_k(G, L^n R_{\mathrm{ab}})$ are periodic groups of finite exponent and that $H_0(G, L^n R_{\mathrm{ab}})$ decomposes into the direct sum of a free abelian group and a periodic group of finite exponent. Moreover, the relevant exponents divide n if $n \geq 3$, and 4 for $n = 2$ [12]. Much more can be said in case when $n = p, p$ a prime. It was proved in [4, Corollary 8.3] that if G has no elements of order p, then there are isomorphisms

$$H_k(G, L^p R_{\mathrm{ab}}) \cong H_{k+4}(G, \mathbb{Z}_p) \quad (k \geq 1) \tag{1.5}$$

where $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ regarded as a trivial G-module, and for $tH_0(G, L^p R_{\mathrm{ab}})$, the torsion subgroup of $H_0(G, L^p R_{\mathrm{ab}})$, we have

$$tH_0(G, L^p R_{\mathrm{ab}}) \cong H_4(G, \mathbb{Z}_p). \tag{1.5}$$

In particular, the torsion subgroup of $F/[\gamma_p R, F]$ can be identified with the homology group $H_4(G, \mathbb{Z}_p)$. The isomorphisms (1.4) and (1.5) are, in fact, a special case of a more general result, namely, a long exact sequence involving $H_k(G, L^p R_{\mathrm{ab}})$ and $tH_0(G, L^p R_{\mathrm{ab}})$ among other things. This long exact sequence [4, Theorem 8.1] was obtained for arbitrary G. In case when G has no p-torsion, some of its members vanish, and we are left with the isomorphisms (1.4) and (1.5). The isomorphism (1.5) has earlier been established in [10].

While (1.4) and (1.5) give a precise description of the torsion in $H_*(G, L^n R_{\mathrm{ab}})$ (and $F/[\gamma_n R, F]$) in case when n is a prime p and G has no p-torsion, not much was known in the case when the degree n is a composite number. In particular, it was not known if non-trivial elements of finite order occur at all in (1.3). In this paper we study the case $n = 4$. Our main result reads as follows.

THEOREM 6.3: *Let G be a group given by a free presentation (1.1). If G has no elements of order 2, then*

$$H_k(G, L^4 R_{\mathrm{ab}}) \cong H_{k+6}(G, \mathbb{Z}_2)$$

for all $k \geq 1$, and

$$tH_0(G, L^4 R_{\mathrm{ab}}) \cong H_6(G, \mathbb{Z}_2).$$

In view of (1.2) this gives:

COROLLARY: *If $G = F/R$ has no 2-torsion, then the torsion subgroup of the quotient $F/[\gamma_4 R, F]$ is isomorphic to $H_6(G, \mathbb{Z}_2)$.*

In particular, for $R = F'$ we get that

$$F/[F', F', F', F', F],$$

the free centre-by-(nilpotent of class 3)-by-abelian group, is torsion-free if $d = \operatorname{rank} F \leq 5$, and that it contains an elementary abelian 2-group of rank $\binom{d}{6}$ when $d \geq 6$. Also, we see that the absence of involutions in G implies that all non-trivial torsion elements in $H_*(G, L^4 R_{\mathrm{ab}})$ have order 2. Hence, the above mentioned bound n for the exponent of $tH_*(G, L^n R_{\mathrm{ab}})$ $(n \geq 3)$ is not optimal in this case. The problem of whether elements of order 4 may occur in $H_*(G, L^4 R_{\mathrm{ab}})$ when G has 2-torsion remains open.

It is interesting to compare Theorem 6.3 with similar results on other polylinear powers of relation modules. For, let $T^n R_{\mathrm{ab}}$, $\Lambda^n R_{\mathrm{ab}}$, $M^n R_{\mathrm{ab}}$ denote the nth tensor power, the nth exterior power, and the nth free metabelian Lie power of R_{ab}, respectively. Then we have for $k \geq 1$ under certain conditions on G, which are given below,

$$H_k(G, T^4 R_{\mathrm{ab}}) \cong H_{k+8} G, \tag{1.6}$$

$$(1.7)\qquad H_k(G,\Lambda^4 R_{\mathrm{ab}}) \cong H_{k+8}(G,\mathbb{Z}_2)\oplus H_{k+7}(G,\mathbb{Z}_3)\oplus H_{k+6}(G,\mathbb{Z}_2),$$

$$(1.8)\qquad H_k(G,M^4 R_{\mathrm{ab}}) \cong H_{k+7}(G,\mathbb{Z}_2)\oplus H_{k+6}(G,\mathbb{Z}_4)\oplus H_{k+4}(G,\mathbb{Z}_2).$$

The isomorphism (1.6) is an easy consequence of MacLane's cup product reduction theorem and does not require any conditions on G. The isomorphism (1.7) was established in [5] under the condition that G has no elements of order 2 and 3, and (1.8) was obtained in [11] for 2-torsion-free G. There are similar results for $k = 0$. Since for prime degree p and G without p-torsion one has $H_k(G, M^p R_{\mathrm{ab}}) \cong H_k(G, L^p R_{\mathrm{ab}}) \cong H_{k+4}(G, \mathbb{Z}_p), k \geq 1$ (see [3]), the difference in the homological behaviour of $M^4 R_{\mathrm{ab}}$ and $L^4 R_{\mathrm{ab}}$ appears quite surprising. Finally, we mention that a complete description of the torsion in $H_k(G, \Lambda^n R_{\mathrm{ab}})$ for arbitrary $k \geq 0, n \geq 2$ and G without $n(n-1)$-torsion has recently been given in [6]. It would be desirable to have a similarly complete result for $H_*(G, L^n R_{\mathrm{ab}})$. This paper is meant as a first step to tackle this problem for non-prime degree n.

The paper is organized as follows. Notation and some preliminary notions are introduced in Section 2. The 4th Lie power $L^4 B$ of a G-module B, which is an extension of a $\mathbb{Z}$-free G-module A by a $\mathbb{Z}$-free G-module C, is examined in Sections 3–5. In particular, in Section 5 we discuss a homomorphism

$$L^4 B \to (C \otimes B) \wedge (C \otimes B),$$

which plays a key role in the whole approach. Finally, in Section 6, we exploit the discussion in the previous sections to examine the homology of G with coefficients in the fourth Lie power of the augmentation ideal IG of $\mathbb{Z}G$ and the relation module R_{ab}. It turns out that $H_k(G, L^4 IG)$ is trivial for $k \geq 1$ and 2-torsion-free G (Theorem 6.2), and this is one of the ingredients of the proof of our main result on $H_*(G, L^4 R_{\mathrm{ab}})$.

2. Preliminaries

In this section we introduce some notation and record some preliminary facts for further reference.

2.1. LIE POWERS. As in Section 1, for a $\mathbb{Z}$-free right G-module A, LA denotes the free Lie ring on A and $L^n A$ denotes the nth Lie power of A. The latter will be regarded as a G-module with diagonal action. Let $\mathcal{A}$ be a free $\mathbb{Z}$-basis of A and assume that $\mathcal{A}$ is totally ordered. Then Hall's basic commutators can be defined in the usual way, and the basic commutators of weight n form a free $\mathbb{Z}$-basis of L^nA (see, e.g., [9, Chapter 5]). In particular, L^4A is as a $\mathbb{Z}$-module freely generated by the basic commutators

$$[u_1, u_2, u_3, u_4], \qquad u_1 > u_2 \leq u_3 \leq u_4, \tag{2.1}$$

$$[[u_1, u_2], [u_3, u_4]], \qquad u_1 > u_2,\ u_3 > u_4,\ [u_1, u_2] > [u_3, u_4] \tag{2.2}$$

($u_1, u_2, u_3, u_4 \in \mathcal{A}$). Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ be subsets of $\mathcal{A}$. By $[\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4]$ we denote the set of all basic commutators (2.1) with $u_i \in \mathcal{A}_i$ ($i = 1, \ldots, 4$), and by $[[\mathcal{A}_1, \mathcal{A}_2], [\mathcal{A}_3, \mathcal{A}_4]]$ we denote the set of all basic commutators (2.2) with $u_i \in \mathcal{A}_i$ ($i = 1, \ldots, 4$). For example, if $\mathcal{A} = \mathcal{A}' \cup \mathcal{A}''$ (disjoint union) and $u' < u''$ for all $u' \in \mathcal{A}'$ and $u'' \in \mathcal{A}''$, we have

$$[\mathcal{A}'', \mathcal{A}', \mathcal{A}', \mathcal{A}''] = \{[u_1'', u_1', u_2', u_2'']; u_1', u_2' \in \mathcal{A}',\ u_1'', u_2'' \in \mathcal{A}'',\ u_1' \leq u_2'\}.$$

Here the conditions $u_1'' > u_1'$, $u_2' \leq u_2''$ are automatically fulfilled. Also, we have $[[\mathcal{A}', \mathcal{A}''], [\mathcal{A}', \mathcal{A}']] = \emptyset$ since there is no basic commutator (2.2) with $u_1 \in \mathcal{A}'$, $u_2 \in \mathcal{A}''$.

Finally, for arbitrary submodules A_1, A_2, A_3, A_4 of A we denote the submodule of L^4A generated by all left normed commutators $[a_1, a_2, a_3, a_4]$ with $a_i \in A_i$ ($i = 1, \ldots, 4$) by $[A_1, A_2, A_3, A_4]$.

2.2. Tensor, exterior and symmetric powers. For A as above, let T^nA denote the nth tensor power

$$T^nA = A \otimes \cdots \otimes A \quad (n \text{ times}),$$

$T^0A = \mathbb{Z}$, and by TA we denote the tensor ring over A. Thus

$$TA = \bigoplus_{n=0}^{\infty} T^nA.$$

The tensor power T^nA will also be regarded as a G-module with diagonal action. It is well-known that TA may be identified with the universal envelope of LA. The restriction of the canonical embedding $LA \to TA$ to L^nA gives an embedding

$$\nu_n \colon L^nA \to T^nA.$$

The tensor power T^nA may also be regarded as a module for S_n, the symmetric group of degree n, by setting

$$(a_1 \otimes \ldots \otimes a_n)\tau^{-1} = a_{1\tau} \otimes \ldots \otimes a_{n\tau} \quad (a_1, \ldots, a_n \in A, \tau \in S_n).$$

We need a special element in the group ring $\mathbb{Z}S_n$. Let $\Omega_1 = 1$ and, for $n > 1$, let

$$\Omega_n = (1-(1,2))(1-(1,2,3))\cdots(1-(1,2,\ldots,n)) \in \mathbb{Z}S_n.$$

The element Ω_n has the useful property

$$\Omega_n^2 = n\Omega. \tag{2.3}$$

Also, a simple induction shows that for $a_1, \ldots, a_n \in A$ one has

$$[a_1, \ldots, a_n]\nu_n = (a_1 \otimes \cdots \otimes a_n)\Omega_n. \tag{2.4}$$

Since the left normed commutators of degree n generate L^nA, the embedding ν_n is completely determined by (2.4). In particular, for $n = 4$, $a, b, c, d \in A$, we have

$$\begin{aligned}[a,b,c,d]\nu_n =& a \otimes b \otimes c \otimes d - b \otimes a \otimes c \otimes d - c \otimes a \otimes b \otimes d + c \otimes b \otimes a \otimes d \\ &- d \otimes a \otimes b \otimes c + d \otimes b \otimes a \otimes c + d \otimes c \otimes a \otimes b - d \otimes c \otimes b \otimes a.\end{aligned}$$

We also have the projection ρ_n: $T^nA \to L^nA$ given by $(a_1 \otimes \ldots \otimes a_n)\rho_n = [a_1, \ldots, a_n]$. The embedding ν_n and the projection ρ_n are related by Wever's formula [9, Chapter 5]

$$\nu_n\rho_n = n, \tag{2.5}$$

i.e. the composite of ν_n and ρ_n amounts to multiplication by n in L^nA. It is well known that the elements

$$u_1\nu_{i_1} \otimes u_2\nu_{i_2} \otimes \cdots \otimes u_m\nu_{i_m},$$

where $u_j \in L_{i_j}A$ are basic commutators such that $u_1 \leq u_2 \leq \cdots \leq u_m$ and $i_1 + i_2 + \cdots + i_m = n$, form a free $\mathbb{Z}$-basis of T^nA [9, Chapter 5]. In particular, the elements

$$u_1 \otimes u_2, \ v\nu_2,$$

where $u_1, u_2 \in \mathcal{A}$ with $u_1 \le u_2$, and v runs over the set of all basic commutators of weight 2, form a free $\mathbb{Z}$-basis of $T^2A = A \otimes A$, and the elements

$$u_1 \otimes u_2 \otimes u_3,\ \ u \otimes v\nu_2, w\nu_3,$$

where $u, u_1, u_2, u_3 \in \mathcal{A}$, $u_1 \le u_2 \le u_3$, and v and w run over the sets of all basic commutators of weight 2 and 3, respectively, form a free $\mathbb{Z}$-basis of T^3A.

Finally, by $\Lambda^n A$ and $S^n A$ we denote the nth exterior and symmetric powers of A. The exterior and symmetric tensors are written as $a_1 \wedge \cdots \wedge a_n$ and $a_1 \circ \cdots \circ a_n$, respectively. For $n = 2$ we also use the notation $\Lambda^2 A = A \wedge A$ for the exterior square, and $S^2A = A \circ A$ for the symmetric square. Note that $L^2A \cong A \wedge A$. Both $\Lambda^n A$ an $S^n A$ will be regarded as G-modules with diagonal action.

2.3 The relation and augmentation sequences. Let G be a group, $\mathbb{Z}G$ the integral group ring of G. The augmentation ideal, that is the kernel of the augmentation map $\mathbb{Z}G \to \mathbb{Z}$, will be denoted by IG. Thus we have a short exact sequence

$$0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

which will be referred to as the augmentation sequence. Now suppose that G is given by a free presentation (1.1), let X be a free generating set for F, and let R_{ab} be the corresponding relation module. Then there is a short exact sequence

$$0 \to R_{ab} \xrightarrow{\mu} P \xrightarrow{\sigma} IG \to 0$$

where P is a free G-module with free generators e_x $(x \in X)$ and the epimorphism σ is determined by $e_x \to (x\pi - 1)$; see, e.g., [1, Chapter 2]. This short exact sequence is usually called the relation sequence stemming from (1.1).

2.4 Homology. Our terminology and notation concerning homology of groups is standard and in line with [1]. For further reference we record some results on the homology of certain G-modules. Let D be a $\mathbb{Z}$-free G-module, and consider the commutative diagram

(2.6)
$$\begin{tikzcd} D \otimes T^n R_{ab} \arrow[r] & D \otimes T^{n-1} R_{ab} \otimes P \\ D \otimes L^n R_{ab} \arrow[u] \arrow[ur] & \end{tikzcd}$$

where the vertical homomorphism is $1 \otimes \nu_n \colon D \otimes L^n R_{\mathrm{ab}} \to D \otimes T^n R_{\mathrm{ab}}$ and the horizontal homomorphism is

$$1 \otimes 1 \otimes \cdots \otimes 1 \otimes \mu \colon D \otimes R_{\mathrm{ab}} \otimes \cdots \otimes R_{\mathrm{ab}} \otimes R_{\mathrm{ab}} \longrightarrow D \otimes R_{\mathrm{ab}} \otimes \cdots \otimes R_{\mathrm{ab}} \otimes P$$

(μ is the embedding from the relation sequence). Similarly, using the inclusion map $IG \to \mathbb{Z}G$, we obtain a commutative diagram

$$\begin{tikzcd} D \otimes T^n IG \arrow[r] & D \otimes T^{n-1} IG \otimes \mathbb{Z}G \\ D \otimes L^n IG \arrow[u] \arrow[ur] & \end{tikzcd} \tag{2.7}$$

Note that $D \otimes T^{n-1} R_{\mathrm{ab}} \otimes P$ and $D \otimes T^{n-1} IG \otimes \mathbb{Z}G$ are free G-modules.

LEMMA 2.1:

(i) *For any $\mathbb{Z}$-free G-module D and $n \geq 3$, $H_k(G, D \otimes L^n R_{\mathrm{ab}})$, $k \geq 1$, and the kernel of the homomorphism*

$$H_0(G, D \otimes L^n R_{\mathrm{ab}}) \to H_0(G, D \otimes T^{n-1} R_{\mathrm{ab}} \otimes P)$$

induced by the diagonal homomorphism in (2.6) are periodic groups of exponent dividing n.

(ii) *For any $\mathbb{Z}$-free G-module D and $n \geq 3$, $H_k(G, D \otimes L^n IG)$, $k \geq 1$, and the kernel of the homomorphism*

$$H_0(G, D \otimes L^n IG) \to H_0(G, D \otimes T^{n-1} IG \otimes \mathbb{Z}G)$$

induced by the diagonal homomorphism in (2.7) are periodic groups of exponent dividing n^2.

(iii) *If G has no elements of order 2, then*

$$H_k(G, (IG \otimes IG) \wedge (IG \otimes IG) \cong H_{k+4}(G, \mathbb{Z}_2)$$

for all $k \geq 1$.

(iv) *If G has no elements of order 2 and A is a free G-module, then $L^4 A$ and $A \wedge A$ are free G-modules as well.*

Proof: For the proof of (i) we refer to [12], Theorem 4.1 and Lemma 4.1. To prove (ii), we apply the homology functor to (2.7):

$$\begin{tikzcd} H_k(G, D \otimes T^n IG) \arrow[r] & H_k(G, D \otimes T^{n-1} IG \otimes \mathbb{Z}G) \\ H_k(G, D \otimes L^n IG) \arrow[u] \arrow[ur] & \end{tikzcd} \tag{2.8}$$

The S_n-action on T^nIG induces an S_n-action on $H_k(G, D \otimes T^nIG)$. By [4, Lemma 5.3], the action of a permutation $\tau \in S_n$ on the kernel of the horizontal homomorphism in (2.8) amounts to multiplication by its sign. In particular, this kernel is annihilated by $\Omega_n \in \mathbb{Z}S_n$. On the other hand, in view of (2.3) and (2.4), Ω_n acts as multiplication by n on the image of $H_k(G, D \otimes L^nIG)$ in $H_k(G, D \otimes T^nIG)$. Now let $a \in H_k(G, D \otimes L^nIG)$ be an element of the kernel of the diagonal in (2.8). Then $aH_k(1 \otimes \nu_n)$ is in the kernel of the horizontal homomorphism. Consider the element $aH_k(1 \otimes \nu_n)\Omega_n H_k(1 \otimes \rho_n)$. Then we have, on the one hand,

$$aH_k(1 \otimes \nu_n)\Omega_n H_k(1 \otimes \rho_n) = 0$$

as $aH_k(1 \otimes \nu_n)$ is annihilated by Ω_n. On the other hand, we have, using the above remark and (2.5),

$$aH_k(1 \otimes \nu_n)\Omega_n H_k(1 \otimes \rho_n) = naH_k(1 \otimes \nu_n\rho_n) = n^2a.$$

Consequently, the kernel of the diagonal in (2.8) is annihilated by n^2, and (ii) follows on noting that $H_k(G, D \otimes T^{n-1}IG \otimes \mathbb{Z}G) = 0$ for $k \geq 1$ as the coefficient module is free.

To show (iii), we note that the tensor square $IG \otimes IG$ may be viewed as a relation module. Indeed, let F be the free group on $\{x_g; g \in G\backslash\{1\}\}$, and consider the presentation of G determined by $x_g \to g$. Then it is easily seen that the free module P from the relation sequence is isomorphic to $IG \otimes \mathbb{Z}G$, and the relation sequence takes the shape

$$0 \to IG \otimes IG \to IG \otimes \mathbb{Z}G \to IG \to 0,$$

that is, the augmentation sequence tensored over $\mathbb{Z}$ on the left with IG. Hence $IG \otimes IG$ is isomorphic to a relation module, and now (iii) follows by (1.4).

Finally, for a proof of (iv) we refer to [10, Theorem 3.11]. ∎

3. A filtration for the fourth Lie power of a module extension

Let G be a group, and let

$$0 \to A \to B \xrightarrow{\beta} C \to 0 \tag{3.1}$$

be a short exact sequence of $\mathbb{Z}$-free G-modules, that is G-modules whose underlying abelian groups are free $\mathbb{Z}$-modules. We will identify A with its image in B.

Let $\mathcal{A}$ and C' be free $\mathbb{Z}$-bases of A and C, respectively. Then B has a free $\mathbb{Z}$-basis $\mathcal{B} = \mathcal{A} \cup \mathcal{C}$ (disjoint union), where $\mathcal{C} = \{v;\ v\beta = v',\ v' \in C'\}$. Assume $\mathcal{A}$ and $\mathcal{C}$ totally ordered and suppose that these orderings are extended to a total ordering of $\mathcal{B}$ by setting $u < v$ for all $u \in \mathcal{A}$, $v \in \mathcal{C}$. Let $\mathcal{L}$ denote the set of all basic commutators of weight 4 defined over $\mathcal{B}$. Thus $\mathcal{L}$ is a free $\mathbb{Z}$-basis of L^4B. In general, the ordering of the basic commutators of weight $n \geq 2$ may be chosen arbitrarily. In this paper, however, we will assume that the basic commutators of weight 2 are ordered in such a way that $[u_1, u_2] < [v, u] < [v_1, v_2]$ for all $u, u_1, u_2 \in \mathcal{A}$ and $v, v_1, v_2 \in \mathcal{C}$. Then the set $\mathcal{L}$ decomposes into the disjoint union

$$\mathcal{L} = \mathcal{L}_0 \cup \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_{10},$$

where

$$\begin{aligned}
\mathcal{L}_0 &= [\mathcal{C},\mathcal{C},\mathcal{C},\mathcal{C}] \cup [[\mathcal{C},\mathcal{C}],[\mathcal{C},\mathcal{C}]],\\
\mathcal{L}_1 &= [\mathcal{C},\mathcal{A},\mathcal{C},\mathcal{C}],\\
\mathcal{L}_2 &= [[\mathcal{C},\mathcal{C}],[\mathcal{C},\mathcal{A}]],\\
\mathcal{L}_3 &= [[\mathcal{C},\mathcal{A}],[\mathcal{C},\mathcal{A}]],\\
\mathcal{L}_4 &= [\mathcal{C},\mathcal{A},\mathcal{A},\mathcal{C}],\\
\mathcal{L}_5 &= [\mathcal{A},\mathcal{A},\mathcal{C},\mathcal{C}],\\
\mathcal{L}_6 &= [[\mathcal{C},\mathcal{C}],[\mathcal{A},\mathcal{A}]],\\
\mathcal{L}_7 &= [\mathcal{C},\mathcal{A},\mathcal{A},\mathcal{A}],\\
\mathcal{L}_8 &= [[\mathcal{C},\mathcal{A}],[\mathcal{A},\mathcal{A}]],\\
\mathcal{L}_9 &= [\mathcal{A},\mathcal{A},\mathcal{A},\mathcal{C}],\\
\mathcal{L}_{10} &= [\mathcal{A},\mathcal{A},\mathcal{A},\mathcal{A}] \cup [[\mathcal{A},\mathcal{A}],[\mathcal{A},\mathcal{A}]].
\end{aligned}$$

Hence L^4B decomposes into the $\mathbb{Z}$-direct sum

$$L^4B = L_0 \dotplus L_1 \dotplus \cdots \dotplus L_{10},$$

where L_k $(0 \leq k \leq 10)$ denotes the $\mathbb{Z}$-span of $\mathcal{L}_k$ in L^4B.

Now let

$$V_k = L_k \dotplus L_{k+1} \dotplus \cdots \dotplus L_{10} \quad (0 \leq k \leq 10).$$

It is easily seen that the V_k 's are G-submodules of L^4B. Thus we have got a G-module filtration

$$L^4B = V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots \supseteq V_{10} \tag{3.2}$$

of L^4B with free abelian quotients. Modulo V_k, the set $\mathcal{L}_{k-1}$ forms a free $\mathbb{Z}$-basis of $V_{k-1}/V_k (k = 1, \ldots, 10)$. Clearly, V_{10} may be identified with the image of L^4A in L^4B under the canonical embedding $L^4A \to L^4B$ induced by the inclusion map $A \to B$. About the quotients of the filtration (3.2) we can say the following.

PROPOSITION 2.1: *There are G-module isomorphisms*

(i)	$V_0/V_1 \cong L^4C$,	(vi)	$V_5/V_6 \cong (A \wedge A) \otimes (C \circ C)$,
(ii)	$V_1/V_2 \cong (C \otimes A \otimes (C \circ C)$,	(vii)	$V_6/V_7 \cong (C \wedge C) \otimes (A \wedge A)$,
(iii)	$V_2/V_3 \cong (C \wedge C) \otimes C \otimes A$,	(viii)	$V_7/V_8 \cong C \otimes S^3A$,
(iv)	$V_3/V_4 \cong (C \otimes A) \wedge (C \otimes A)$,	(ix)	$V_8/V_9 \cong C \otimes A \otimes (A \wedge A)$,
(v)	$V_4/V_5 \cong C \otimes (A \circ A) \otimes C$,	(x)	$V_9/V_{10} \cong L^3A \otimes C$.

Proof: Let $a_1, a_2, a_3, a_4 \in A, c_1, c_2, c_3, c_4 \in C$, and let $b_1, b_2, b_3, b_4 \in B$ such that $b_i\beta = c_i$ $(1 \le i \le 4)$. The isomorphisms (i)-(x) are determined by the following maps:

(i) $$[c_1, c_2, c_3, c_4] \to [b_1, b_2, b_3, b_4] + V_1,$$

(ii) $$c_1 \otimes a_1 \otimes (c_2 \circ c_3) \to [b_1, a_1, b_2, b_3] + V_2,$$

(iii) $$(c_1 \wedge c_2) \otimes c_3 \otimes a_1 \to [[b_1, b_2], [b_3, a_1]] + V_3,$$

(iv) $$(c_1 \otimes a_1) \wedge (c_2 \otimes a_2) \to [[b_1, a_1], [b_2, a_2]] + V_4,$$

(v) $$c_1 \otimes (a_1 \circ a_2) \otimes c_2 \to [b_1, a_1, a_2, b_2] + V_5,$$

(vi) $$(a_1 \wedge a_2) \otimes (c_1 \circ c_2) \to [a_1, a_2, c_1, c_2] + V_6,$$

(vii) $$(c_1 \wedge c_2) \otimes (a_1 \wedge a_2) \to [[b_1, b_2], [a_1, a_2]] + V_7,$$

(viii) $$c_1 \otimes (a_1 \circ a_2 \circ a_3) \to [b_1, a_1, a_2, a_3] + V_8,$$

(ix) $$c_1 \otimes a_1 \otimes (a_2 \wedge a_3) \to [[b_1, a_1], [a_2, a_3]] + V_9,$$

(x) $$[a_1, a_2, a_3] \otimes c_1 \to [a_1, a_2, a_3, b_1] + V_{10}.$$

It is easy to check that these maps are correctly defined and that they define indeed G-isomorphisms. We give a proof for (vi), and leave the rest to the reader. To verify (vi), note that the map

$$a_1 \otimes a_2 \otimes c_1 \otimes c_2 \to [a_1, a_2, b_1, b_2] + V_6$$

defines a homomorphism

$$A \otimes A \otimes C \otimes C \to V_5/V_6. \tag{3.3}$$

First of all, the map is correctly defined as if b_1' and b_2' are elements of B with $b_1'\beta = c$, and $b_2'\beta = c_2$, then $b_1 = b_1' + a'$ and $b_2 = b_2' + a''$ for some $a', a'' \in A$. Hence

$$\begin{aligned}[a_1, a_2, b_1, b_2] &= [a_1, a_2, b_1' + a', b_2' + a''] \\ &= [a_1, a_2, b_1', b_2'] + [a_1, a_2, a', b_2'] + [a_1, a_2, b_1', a''] + [a_1, a_2, a', a''] \\ &\equiv [a_1, a_2, b_1', b_2'] \bmod V_6\end{aligned}$$

as $[a_1, a_2, a', b_2'] \in V_9, [a_1, a_2, b_1', a''] \in V_7$ and $[a_1, a_2, a_1', a''] \in V_{10}$. This shows that our map does not depend on the particular choice of the b's as inverse images of the c's under the epimorphism β. Using the anticommutativity law and the Jacobi identity we have

$$[a_1, a_2, b_1, b_2] = -[a_2, a_1, b_1, b_2]$$

and

$$\begin{aligned}[a_1, a_2, b_1, b_2] &= [a_1, a_2, b_2, b_1] - [[b_1, b_2], [a_1, a_2]] \\ &\equiv [a_1, a_2, b_2, b_1] \bmod V_6\end{aligned}$$

as $[[b_1, b_2], [a_1, a_2]] \in V_6$. Hence the homomorphism (3.3) factors through $(A \wedge A) \otimes (C \circ C)$ giving a G-homomorphism

$$(A \wedge A) \otimes (C \circ C) \to V_5/V_6. \tag{3.4}$$

Now $(A \wedge A) \otimes (C \circ C)$ has a free $\mathbb{Z}$-basis consisting of all elements $(u_1 \wedge u_2) \otimes (v_1' \circ v_2')$ where $u_1, u_2 \in \mathcal{A}$, $u_1 > u_2$, $v_1', v_2' \in C'$, $v_1' \leq v_2'$, and it remains to note that the G-homomorphism (3.4) maps this basis one-to-one onto $\mathcal{L}_5 + V_6$, a free $\mathbb{Z}$-basis of V_5/V_6. Hence (3.4) is bijective and therefore an isomorphism.

4. Sections and submodules of L^4B

Let B as in Section 3, and consider the submodules $[A, A, A, B]$ and $[B, A, A, B]$ of L^4B. It is easily seen that these submodules coincide with V_9 and V_4, respectively:

$$[A, A, A, B] = V_9, \quad [B, A, A, B] = V_4.$$

Now consider the homomorphism

$$L^4B \to C \otimes B \otimes B \otimes C \tag{4.1}$$

defined as the composite of

$$\nu_4\colon L^4B \to B \otimes B \otimes B \otimes B$$

and

$$\beta \otimes 1 \otimes 1 \otimes \beta\colon B \otimes B \otimes B \otimes B \to C \otimes B \otimes B \otimes C.$$

Hence for $b_1, b_2, b_3, b_4 \in B$ we have

$$[b_1, b_2, b_3, b_4] \to b_1\beta \otimes b_2 \otimes b_3 \otimes b_4\beta - b_2\beta \otimes b_1 \otimes b_3 \otimes b_4\beta - \cdots$$

Clearly, if $a_1, a_2 \in A = \ker \beta$, then the homomorphism (4.1) maps the left normed commutator $[b_1, a_1, a_2, b_2] \in [B, A, A, B]$ to $b_1\beta \otimes a_1 \otimes a_2 \otimes b_2\beta$. Hence the restriction of (4.1) to $[B, A, A, B]$ maps this submodule onto $C \otimes A \otimes A \otimes C \subseteq C \otimes B \otimes B \otimes C$. Consequently, the map

$$[b_1, a_1, a_2, b_2] \to b_1\beta \otimes a_1 \otimes a_2 \otimes b_2\beta$$

defines a G-homomorphism

$$\psi\colon [B, A, A, B] \to C \otimes A \otimes A \otimes C.$$

LEMMA 4.2: $\operatorname{Ker} \psi = V_7$.

Proof: One has clearly $V_7 \subseteq \ker \psi$. The inverse inclusion will be proved once we show that ψ induces an isomorphism $V_4/V_7 \longrightarrow C \otimes A \otimes A \otimes C$. But this is the case since ψ maps $\mathcal{L}_4 \cup \mathcal{L}_5 \cup \mathcal{L}_6$ one-to-one onto a free $\mathbb{Z}$-basis of $C \otimes A \otimes A \otimes C$. Indeed, for $v_1, v_2 \in \mathcal{C}, u_1, u_2 \in \mathcal{A}$, and the basis elements of $\mathcal{L}_4, \mathcal{L}_5$ and $\mathcal{L}_6$, respectively, we have

$$[v_1, u_1, u_2, v_2]\psi = v_1' \otimes u_1 \otimes u_2 \otimes v_2', \tag{4.3}$$

$$\begin{aligned}[u_1, u_2, v_1, v_2]\psi &= (-[v_1, u_1, u_2, v_2] + [v_1, u_2, u_1, v_2])\psi \\ &= -v_1' \otimes u_1 \otimes u_2 \otimes v_2' + v_1' \otimes u_2 \otimes u_1 \otimes v_2' \\ &= -v_1' \otimes [u_1, u_2]\nu_2 \otimes v_2',\end{aligned} \tag{4.4}$$

$$\begin{aligned}[[v_1, v_2], [u_1, u_2]]\psi =&([v_1, u_1, u_2, v_2] - [v_1, u_2, u_1, v_2] \\ &- [v_2, u_1, u_2, v_1] + [v_2, u_2, u_1, v_1])\psi \\ =&v_1' \otimes u_1 \otimes u_2 \otimes v_2' - v_1' \otimes u_2 \otimes u_1 \otimes v_2' \\ &- v_2' \otimes u_1 \otimes u_2 \otimes v_1' - v_2' \otimes u_1 \otimes u_2 \otimes v_1' \\ =&v_1' \otimes [u_1, u_2]\nu_2 \otimes v_2' - v_2' \otimes [u_1, u_2]\nu_2 \otimes v_1',\end{aligned} \tag{4.5}$$

where in (4.3) we have $u_1 \leq u_2$ with arbitrary $v_1, v_2 \in \mathcal{C}$, in (4.4) we have $u_1 > u_2$ and $v_1 \leq v_2$, and in (4.5) we have $v_1 > v_2$ and $u_1 > u_2$. It remains to note the elements (4.3), (4.4) and (4.5) form indeed a free $\mathbb{Z}$-basis of $C \otimes A \otimes A \otimes C$. ∎

Now let $a_1, a_2 \in A$, $c_1 \in C$, $b_1 \in B$ such that $b_1\beta = c_1$, and let $b_2 \in B$. We claim that the map

$$c_1 \otimes a_1 \otimes a_2 \otimes b_2 \to [b_1, a_1, a_2, b_2] + V_9$$

defines a G-homomorphism

$$\theta\colon C \otimes A \otimes A \otimes B \to V_4/V_9.$$

Indeed, we only need to check that the map is correctly defined. But this is clear since if $b_1' \in B$ is another inverse image of c_1 under β, then $b_1 = b_1' + a'$ for some $a' \in A$, so

$$\begin{aligned}[b_1, a_1, a_2, b_2] &= [b_1' + a', a_1, a_2, b_2] \\ &= [b_1', a_1, a_2, b_2] + [a', a_1, a_2, b_2] \\ &= [b_1', a_1, a_2, b_2] \bmod V_9\end{aligned}$$

as $[a', a_1, a_2, b_2] \in V_9$. Let $\tilde{\theta}$ denote the restriction of θ to the submodule $C \otimes A \otimes A \otimes A \subseteq C \otimes A \otimes A \otimes B$, i.e. for $c \in C$ and $a_1, a_2, a_3 \in A$ we have $(c \otimes a_1 \otimes a_2 \otimes a_3)\tilde{\theta} = [b, a_1, a_2, a_3] + V_9$ where $b \in B$ with $b\beta = c$. Now return to the homomorphism ψ. Since $V_9 \subseteq \ker\psi$, ψ factors through V_4/V_9, i.e. it determines a homomorphism $\overline{\psi}\colon V_4/V_9 \to C \otimes A \otimes A \otimes C$. It is readily seen that

(4.6)
$$\begin{tikzcd} 0 \arrow[r] & V_7/V_9 \arrow[r] & V_4/V_9 \arrow[r, "\overline{\psi}"] & C \otimes A \otimes A \otimes C \arrow[r] & 0 \\ 0 \arrow[r] & C \otimes A \otimes A \otimes A \arrow[r] \arrow[u, "\tilde{\theta}"] & C \otimes A \otimes A \otimes B \arrow[r, "\lambda"] \arrow[u, "\theta"] & C \otimes A \otimes A \otimes C \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

with $\lambda = 1 \otimes 1 \otimes 1 \otimes \beta$ is a commutative diagram with exact rows.

LEMMA 4.2: *There is an exact sequence*

$$0 \longrightarrow C \otimes L^3 A \xrightarrow{1\otimes\nu_3} C \otimes A \otimes A \otimes A \xrightarrow{\tilde{\theta}} V_7/V_9 \to 0.$$

Proof: For a_1, a_2, a_3, c, b as above we get, by using the definitions of ν_3 and $\tilde{\theta}$, as well as the Jacobi identity and the anticommutativity law,

$$\begin{aligned}
&(c \otimes a_1 \otimes a_2 \otimes a_3)(1 \otimes \nu_3)\tilde{\theta} \\
&\qquad = [b, a_1, a_2, a_3] - [b, a_2, a_1, a_3] - [b, a_3, a_1, a_2] + [b, a_3, a_2, a_1] + V_9 \\
&\qquad = -[a_1, a_2, b, a_3] - [[b, a_3], [a_1, a_2]] + V_9 \\
&\qquad = -[a_1, a_2, a_3, b] + V_9 \\
&\qquad = 0 + V_9
\end{aligned}$$

as $[a_1, a_2, a_3, b] \in V_9$. Hence the image of $1 \otimes \nu_3$ is in the kernel of $\tilde{\theta}$. On the other hand, $C \otimes A \otimes A \otimes A$ has a free $\mathbb{Z}$-basis consisting of all elements

$$v' \otimes u_1 \otimes u_2 \otimes u_3, \qquad u_1 \le u_2 \le u_3, \tag{4.7}$$

$$v' \otimes [u_2, u_3], \qquad u_2 > u_3, \tag{4.8}$$

$$v' \otimes [u_1, u_2, u_3] v_3, \qquad u_1 > u_2 \le u_3, \tag{4.9}$$

where $v' \in C'$, $u_1, u_2, u_3 \in \mathcal{A}$ (see Section 2). The elements (4.9) are a free $\mathbb{Z}$-basis of the image of $1 \otimes \nu_3$. For the elements (4.7) and (4.8) we have

$$(v' \otimes u_1 \otimes u_2 \otimes u_3)\tilde{\theta} = [v, u_1, u_2, u_3] + V_9,$$

$$\begin{aligned}
(v' \otimes u_1 \otimes [u_2, u_3]\nu_2)\tilde{\theta} &= [v, u_1, u_2, u_3] - [v, u_1, u_3, u_2] + V_9 \\
&= [[v_1, u_1], [u_2, u_3]] + V_9.
\end{aligned}$$

Hence $\tilde{\theta}$ maps the elements (4.7) one-to-one onto $\mathcal{L}_7 + V_9$, and the elements (4.8) one-to-one onto $\mathcal{L}_8 + V_9$. Therefore, $\operatorname{Ker} \tilde{\theta} = \operatorname{Im}(1 \otimes \nu_3)$, and this completes the proof of the lemma. ∎

In view of the commutative diagram (4.6), we can now state the following.

Corollary 4.3: *There is a short exact sequence*

$$0 \to C \otimes L^3 A \to C \otimes A \otimes A \otimes B \to V_4/V_9 \to 0$$

of G-modules.

We conclude this section with

Lemma 4.4: *The map*

$$c_1 \otimes a \otimes c_2 \otimes c_3 \to [b_1, a, b_2, b_3] + V_3,$$

where $a \in A$, $c_1, c_2, c_3 \in C$ and $b_1, b_2, b_3 \in B$ such that $b_i\beta = c_i$ $(i = 1,2,3)$, determines an isomorphism $C \otimes A \otimes C \otimes C \longrightarrow V_1/V_3$.

Proof: It is easily checked that the above map is correctly defined, thus giving a G-homomorphism $C \otimes A \otimes C \otimes C \to V_1/V_3$. The tensor product $C \otimes A \otimes C \otimes C$ has a $\mathbb{Z}$-basis consisting of all elements

$$v_1' \otimes u \otimes v_2' \otimes v_3', \quad u \in \mathcal{A}, \quad v_1', v_2', v_3' \in \mathcal{C}' \quad \text{and} \quad v_2' \leq v_3'$$
$$\text{and} \quad v_1' \otimes u \otimes [v_2', v_3']\nu_2, \quad u \in \mathcal{A}, \quad v_1', v_2', v_3' \in \mathcal{C}' \quad \text{and} \quad v_2' > v_3'.$$

To show that our homomorphism is bijective, it remains to note that it maps this free $\mathbb{Z}$-basis one-to-one onto $(\mathcal{L}_1 \cup \mathcal{L}_2) + V_3$. Indeed

$$v_1' \otimes u \otimes v_2' \otimes v_3' \to [v_1, u, v_2, v_3] + V_3$$

and

$$v_1' \otimes u \otimes [v_2', v_3']\nu_2 = v_1' \otimes u \otimes v_2' \otimes v_3' - v_1' \otimes u \otimes v_3' \otimes v_2'$$

is mapped to

$$[v_1, u, v_2, v_3] - [v_1, u, v_3, v_2] + V_3 = [[v_2, v_3], [v_1, u]] + V_3. \qquad \blacksquare$$

5. The homomorphism $L^4B \to (C \otimes B) \wedge (C \otimes B)$

In this section we define and examine a homomorphism mapping the fourth Lie power L^4B into the exterior square $(C \otimes B) \wedge (C \otimes B)$. We start with an embedding of the fourth Lie power into the exterior square of a tensor square.

Let A be a $\mathbb{Z}$-free G-module, and consider L^4A. Let $\tilde{\gamma}$: $L^4A \to (A \otimes A) \wedge (A \otimes A)$ be the composite of the embedding ν_4: $L^4A \to T^4A$, the automorphism $(1,2)$: $T^4A \to T^4A$ determined by the action of the transposition $(1,2) \in S_4$, i.e. for $a, b, c, d \in A$ we have $(a \otimes b \otimes c \otimes d)(1,2) = b \otimes a \otimes c \otimes d$, and the canonical epimorphism $T^4A \to (A \otimes A) \wedge (A \otimes A)$ given by $a \otimes b \otimes c \otimes d \to (a \otimes b) \wedge (c \otimes d)$. An easy calculation shows that

$$\begin{aligned}[a, b, c, d]\tilde{\gamma} = 2(b \otimes a) \wedge (c \otimes d) - 2(a \otimes b) \wedge (c \otimes d) \\ - 2(a \otimes c) \wedge (b \otimes d) + 2(b \otimes c) \wedge (a \otimes d).\end{aligned}$$

Hence the map

$$[a, b, c, d] \to (b \otimes a) \wedge (c \otimes d) - (a \otimes b) \wedge (c \otimes d) - (a \otimes c) \wedge (b \otimes d) + (b \otimes c) \wedge (a \otimes d)$$

defines a G-homomorphism

$$\gamma\colon L^4A \to (A\otimes A)\wedge(A\otimes A).$$

LEMMA 5.1: *The homomorphism γ is an embedding.*

Proof: Let $\delta\colon (A\otimes A)\wedge(A\otimes A)\to L^4A$ be the composite of the homomorphism $(A\otimes A)\wedge(A\otimes A)\to T^4A$ defined by $(a\otimes b)\wedge(c\otimes d)\to a\otimes b\otimes c\otimes d - c\otimes d\otimes a\otimes b$, the automorphism $(1,2)\colon T^4A\to T^4A$, and the canonical projection $\rho_4\colon T^4A\to L^4A$. A straightforward calculation using Wever's formula (2.5) with $n=4$ shows that the composite of γ and δ amounts to multiplication by 4 in L^4A: $\gamma\delta=4$. Since L^4A is free abelian, $\gamma\delta$ is injective. Consequently, γ is an embedding. ∎

Now let B be as in Sections 3 and 4, and let

$$\varphi\colon L^4B\to(C\otimes B)\wedge(C\otimes B)$$

denote the composite of $\gamma\colon L^4B\to(B\otimes B)\wedge(B\otimes B)$ and the epimorphism

$$(\beta\otimes 1)\wedge(\beta\otimes 1)\colon (B\otimes B)\wedge(B\otimes B)\to(C\otimes B)\wedge(C\otimes B).$$

Hence for $a,b,c,d\in B$ we have

$$\begin{aligned}[a,b,c,d]\varphi = (b\beta\otimes a)\wedge(c\beta\otimes d) - (a\beta\otimes b)\wedge(c\beta\otimes d)\\ - (a\beta\otimes c)\wedge(b\beta\otimes d) + (b\beta\otimes c)\wedge(a\beta\otimes d).\end{aligned}$$

LEMMA 5.2:

(i) $\operatorname{Ker}\varphi = [B,A,A,B]$

(ii) $\operatorname{Coker}\varphi\cong(C\otimes C)\wedge(C\otimes C)/(L^4C)\gamma$.

Proof: One has clearly from the definition that $[B,A,A,B]=V_4$ is contained in the kernel of φ. To prove the lemma it is therefore sufficient to show that φ induces an embedding $L^4B/V_4 \to (C\otimes B)\wedge(C\otimes B)$ with cokernel $(C\otimes C)\wedge(C\otimes C)/(L^4C)\gamma$. In view of the short exact sequence

$$0\to C\otimes A\to C\otimes B\to C\otimes C\to 0,$$

the exterior square $(C\otimes B)\wedge(C\otimes B)$ has a filtration

$$(C\otimes B)\wedge(C\otimes B) = W_0\supseteq W_1\supseteq W_2$$

where $W_0/W_1 \cong (C \otimes C) \wedge (C \otimes C)$, $W_1/W_2 \cong C \otimes A \otimes C \otimes C$, and W_2 is the canonical image of $(C \otimes A) \wedge (C \otimes A)$ in $(C \otimes B) \wedge (C \otimes B)$. We claim the φ induces isomorphisms

$$V_3/V_4 \longrightarrow W_2, \quad V_1/V_2 \longrightarrow W_1/W_2,$$

and an embedding

$$V_0/V_1 \rightarrow W_0/W_1.$$

Indeed, for $[[v_1, u_1], [v_2, u_2]] = [v_1, u_1, v_2, u_2] - [v_1, u_1, u_2, v_2] \in \mathcal{L}_3$ we have

$$[[v_1, u_1], [v_2, u_2]]\varphi = (v_1' \otimes u_1) \wedge (v_2' \otimes u_2).$$

Hence φ maps V_3/V_4 isomorphically onto $(C \otimes A) \wedge (C \otimes A)$. By Lemma 4.4, we have an isomorphism $C \otimes A \otimes C \otimes C \longrightarrow V_1/V_3$ given by

$$c_1 \otimes a \otimes c_2 \otimes c_3 \rightarrow [b_1, a, b_2, b_3] + V_3$$

where $a \in A$, $b_i \in B$ and $c_i = b_i\beta$ $(i = 1, 2, 3)$. But

$$[b_1, a, b_2, b_3]\varphi = (c_1 \otimes a) \wedge (c_2 \otimes b_3) \equiv (c_1 \otimes a) \wedge (c_2 \otimes c_3) \bmod W_2.$$

Hence φ induces an isomorphism $V_1/V_3 \longrightarrow W_1/W_2$. Finally, we have the commutative diagram

$$\begin{array}{ccc} L^4C & \xrightarrow{\gamma} & (C \otimes C) \wedge (C \otimes C) \\ \uparrow & & \uparrow{\scriptstyle (1\otimes\beta)\wedge(1\otimes\beta)} \\ L^4B & \xrightarrow{\varphi} & (C \otimes B) \wedge (C \otimes B) \end{array}$$

in which the left vertical homomorphism is the canonical projection $L^4B \rightarrow L^4C$ induced by β: $B \rightarrow C$. This shows that φ induces the embedding γ on the top quotients $V_0/V_1 \cong L^4C$ and $W_0/W_1 \cong (C \otimes C) \wedge (C \otimes C)$ of our filtrations of L^4B and $(C \otimes B) \wedge (C \otimes B)$, respectively. Hence, φ induces an embedding of L^4B/V_4 into $(C \otimes B) \wedge (C \otimes B)$, and the cokernel of φ is isomorphic to the cokernel of γ: $L^4C \rightarrow (C \otimes C) \wedge (C \otimes C)$. This completes the proof of the lemma. ∎

Corollary 5.3: *There is a 4-term exact sequence*

$$0 \rightarrow V_4 \rightarrow L^4B \rightarrow (C \otimes B) \wedge (C \otimes B) \rightarrow \operatorname{Coker}\varphi \rightarrow 0.$$

6. Homology of fourth Lie powers

In this final section we exploit our discussion of L^4B to compute the homology of G with coefficients in L^4IG and $L^4R_{\rm ab}$. By Lemma 2.1(i),(ii), the homology groups with coefficients in $L^4R_{\rm ab}$ and L^4IG in positive dimensions as well as the torsion subgroup of $H_0(G, L^4R_{\rm ab})$ are 2-groups. Hence we can localize at 2 and work over $\mathbb{Z}_{(2)}$, the ring of 2-adic integers, instead of $\mathbb{Z}$, as $H_k(G, D \otimes \mathbb{Z}_{(2)}) \cong H_k(G, D) \otimes \mathbb{Z}_{(2)}$ for any G-module D. We use "^" to denote localized objects, i.e. we write $\hat{B}$, $\hat{R}_{\rm ab}$, $I\hat{G}, \ldots$ instead of $B \otimes \mathbb{Z}_{(2)}$, $R_{\rm ab} \otimes \mathbb{Z}_{(2)}$, $\mathrm{IG} \otimes \mathbb{Z}_{(2)}, \ldots$ Suppose we are given a short exact sequence (3.1) of $\mathbb{Z}$-free G-modules. Then we have the following exact sequences:

$$0 \to L^4\hat{A} \to \hat{V}_9 \to L^3\hat{A} \otimes \hat{C} \to 0, \tag{6.1}$$

$$0 \to \hat{C} \otimes L^3\hat{A} \to \hat{C} \otimes \hat{A} \otimes \hat{A} \otimes \hat{B} \to \hat{V}_4/\hat{V}_9 \to 0, \tag{6.2}$$

$$0 \to \hat{V}_9 \to \hat{V}_4 \to \hat{V}_4/\hat{V}_9 \to 0, \tag{6.3}$$

$$0 \to \hat{V}_4 \to L^4\hat{B} \xrightarrow{\hat{\varphi}} (\hat{C} \otimes \hat{B}) \wedge (\hat{C} \otimes \hat{B}) \to \operatorname{Coker} \hat{\varphi} \to 0. \tag{6.4}$$

The sequence (6.1) comes immediately from Proposition 3.1, (6.2) is the localized version of the sequence in Corollary 4.3, (6.3) is obvious, and (6.4) is the localized version of the 4-term sequence in Corollary 5.3. The following lemma might appear rather clumsy. It provides, however, a unified approach towards $H_*(G, L^4R_{\rm ab})$ and $H_*(G, L^4\mathrm{IG})$.

LEMMA 6.1: *Suppose we are given an exact sequence (3.1) such that the following four conditions hold.*

(i) *B is a free G-module.*
(ii) *G has no elements of order 2.*
(iii) *$H_k(G, \hat{C} \otimes L^3\hat{A}) = 0$ for all $k \geq 1$, and the homomorphism $H_0(G, \hat{C} \otimes L^3\hat{A}) \to H_0(G, \hat{C} \otimes \hat{A} \otimes \hat{A} \otimes \hat{B})$ induced by the embedding in (6.2) is injective.*
(iv) *$H_k(G, \operatorname{Coker} \hat{\varphi})$ is a torsion group for all $k \geq 2$.*

Then

$$H_k(G, L^4\hat{A}) \cong H_{k+2}(G, \operatorname{Coker} \hat{\varphi})$$

for all $k \geq 1$. *If, in addition to (i) – (iv), we have that*

(v) $H_0(G, \hat{V}_4/\hat{V}_9)$ *is torsion-free,*

then

$$tH_0(G, L^4\hat{A}) \cong H_2(G, \operatorname{Coker}\hat{\varphi}).$$

Proof: Since B is a free G-module, $\hat{C} \otimes \hat{A} \otimes \hat{A} \otimes \hat{B}$ is a free $\mathbb{Z}_{(2)}G$-module. In particular, it has trivial homology in all positive dimensions and $H_0(G, \hat{C} \otimes \hat{A} \otimes \hat{A} \otimes \hat{B})$ is a free $\mathbb{Z}_{(2)}$-module. Now apply the homology functor to (6.2). Then the long exact homology sequence implies, in view of the freeness of the middle term and condition (iii), that

$$H_k(G, \hat{V}_4/\hat{V}_9) = 0 \tag{6.5}$$

for all $k \geq 1$. Note also that $H_0(G, \hat{C} \otimes L^3\hat{A}) \cong H_0(G, L^3\hat{A} \otimes \hat{C})$ is torsion-free. Now apply the homology functor to (6.1) and consider the long exact homology sequence. Then (iii) and the fact the $H_0(G, L^3\hat{A} \otimes \hat{C})$ is torsion-free yield isomorphisms

$$H_k(G, L^3\hat{A}) \cong H_k(G, \hat{V}_9), \quad k \geq 1, \tag{6.6}$$

$$tH_0(G, L^4\hat{A}) \cong \mathrm{tH}_0(G, \hat{V}_9). \tag{6.7}$$

Now turn to (6.3) and apply the homology functor. Then the long exact homology sequence implies, in view of (6.5), that there are isomorphisms

$$H_k(G, \hat{V}_9) \cong H_k(G, \hat{V}_4), \quad k \geq 1. \tag{6.8}$$

Moreover, if $H_0(G, \hat{V}_4/\hat{V}_9)$ is torsion-free, that is condition (v) holds, we also have that

$$tH_0(G, \hat{V}_9) \cong tH_0(G, \hat{V}_4). \tag{6.9}$$

Now we turn to (6.4). Since G has no 2-torsion and B is G-free, Lemma 2.1(iv) implies that $L^4\hat{B}$ and $(\hat{C} \otimes \hat{B}) \wedge (\hat{C} \otimes \hat{B})$ are free $\mathbb{Z}_{(2)}G$-modules. Hence we get from (6.4) by dimension shifting

$$H_k(G, \hat{V}_4) \cong H_{k+2}(G, \operatorname{Coker}\hat{\varphi}), \quad k \geq 1, \tag{6.10}$$

and, using condition (iv),

$$tH_0(G, \hat{V}_4) \cong H_2(G, \operatorname{Coker}\hat{\varphi}). \tag{6.11}$$

Now the assertion of the lemma follows by combining (6.6) with (6.8) and (6.10) for the case $k \geq 1$, and by combining (6.7) with (6.9) and (6.11) for the case $k = 0$. ∎

Now we deduce our result about L^4IG.

THEOREM 6.2: *Let G be a group without 2-torsion. Then $H_k(G, L^4 IG) = 0$ for all $k \geq 1$.*

Proof: We need to show that $H_k(G, L^4 I\hat{G}) = 0$ for all $k \geq 1$. For, we check that the conditions (i)–(iv) of Lemma 6.1 hold for the augmentation sequence

$$0 \to \mathrm{IG} \to \mathbb{Z}G \to \mathbb{Z} \to 0.$$

This is clear for (i) and (ii), and (iii) holds by Lemma 2.1(ii) as 3-groups vanish under localization at 2. Moreover, we have $\mathrm{Coker}\,\hat{\varphi} = 0$ as $\mathrm{Coker}\,\hat{\varphi} \cong (\mathbb{Z}_{(2)} \otimes \mathbb{Z}_{(2)}) \wedge (\mathbb{Z}_{(2)} \otimes \mathbb{Z}_{(2)})/(L^4\mathbb{Z}_{(2)})\gamma$ by Lemma 5.2, but $\mathbb{Z}_{(2)} \otimes \mathbb{Z}_{(2)} \cong \mathbb{Z}_{(2)}$, so its exterior square is trivial. In particular, condition (iv) holds. Now the theorem follows by Lemma 6.1. ∎

We conclude the paper with the proof of our main result, Theorem 6.3, which is stated in Section 1.

Proof of Theorem 6.3: It is sufficient to examine $H_*(G, L^4\hat{R}_{\mathrm{ab}})$. We claim that conditions (i)-(v) of Lemma 6.1 hold for the relation sequence

$$0 \to R_{\mathrm{ab}} \to P \to \mathrm{IG} \to 0.$$

This is clear again for (i) and (ii), and (iii) holds by Lemma 2.1(i). By Lemma 5.2 we have a short exact sequence

$$(6.12) \qquad 0 \to L^4 I\hat{G} \to (I\hat{G} \otimes I\hat{G}) \wedge (I\hat{G} \otimes I\hat{G}) \to \mathrm{Coker}\,\hat{\varphi} \to 0.$$

By Theorem 6.1, $H_k(G, L^4 I\hat{G}) = 0$ for all $k \geq 1$. Hence the long exact homology sequence determined by (6.12) yields isomorphisms

$$(6.13) \qquad H_k(G, \mathrm{Coker}\,\hat{\varphi}) \cong H_k(G, (I\hat{G} \otimes I\hat{G}) \wedge (I\hat{G} \otimes I\hat{G})),$$

for all $k \geq 2$. Lemma 2.1(iii) tells us that

$$(6.14) \qquad H_k(G, (I\hat{G} \otimes I\hat{G}) \wedge (I\hat{G} \otimes I\hat{G})) \cong H_{k+4}(G, \mathbb{Z}_2)$$

for all $k \geq 1$. In particular, condition (iv) of Lemma 6.1 is fulfilled. Finally, to verify condition (v), note that (6.2) implies that

$$\hat{V}_4/\hat{V}_9 \cong I\hat{G} \otimes (\hat{R}_{\mathrm{ab}} \otimes R_{\mathrm{ab}} \otimes P/(L^3\hat{R}_{\mathrm{ab}})\hat{\nu}_3(1 \otimes 1 \otimes \hat{\mu})). \tag{6.15}$$

Put $\hat{Q} = \hat{R}_{\mathrm{ab}} \otimes R_{\mathrm{ab}} \otimes \hat{P}/(L^3 R_{\mathrm{ab}})\hat{\nu}_3(1 \otimes 1 \otimes \hat{\mu})$. Then the right hand side of (6.15) fits into the short exact sequence

$$0 \to I\hat{G} \otimes \hat{Q} \to \mathbb{Z}_{(2)}G \otimes \hat{Q} \to \hat{Q} \to 0, \tag{6.16}$$

and $\hat{Q}$ itself fits into the short exact sequence

$$0 \to L^3\hat{R}_{\mathrm{ab}} \to \hat{R}_{\mathrm{ab}} \otimes \hat{R}_{\mathrm{ab}} \otimes \hat{P} \to \hat{Q} \to 0. \tag{6.17}$$

In view of Lemma 2.1(i), the long exact homology sequence determined by (6.17) implies that $H_k(G, \hat{Q}) = 0$ for all $k \geq 1$. Now the long exact homology sequence determined by (6.16) yields that $H_0(G, I\hat{G} \otimes \hat{Q}) \cong H_0(G, \hat{V}_4/\hat{V}_9)$ is torsion-free as it is embedded in the free $\mathbb{Z}_{(2)}$-module $H_0(G, \mathbb{Z}_{(2)}G \otimes \hat{Q})$. This means we can apply Lemma 6.1 to the relation sequence. Hence there are isomorphisms

$$\begin{aligned} H_k(G, L^4\hat{R}_{\mathrm{ab}}) &\cong H_{k+2}(G, \operatorname{Coker}\hat{\varphi}) \quad (k \geq 1), \\ tH_0(G, L^4\hat{R}_{\mathrm{ab}}) &\cong H_2(G, \operatorname{Coker}\hat{\varphi}). \end{aligned}$$

Now the theorem follows by combining these with (6.13) and (6.14). ∎

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